Application of the Coupled Cluster Method to a Hamiltonian Lattice Field Theory

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Abstract

The coupled cluster method has been applied to the eigenvalue problem lattice Hamiltonian QCD (without quarks) for SU(2) gauge fields in two space dimensions.

Using a recently presented new formulation and the truncation prescription of Guo et al. we were able to compute the ground state and the lowest 0^+-glueball mass up to the sixth order of the coupled cluster expansion.

The results show evidence for a "scaling window" (i.e. good convergence and constance of dimensionless quantities) around $\beta = 4/g^2 \approx 3$.

A comparison of our results to those of other methods is presented.

1 Introduction

It is the purpose of this paper to present our first results of an attempt to compute the spectrum of a lattice gauge field theory within the Hamiltonian formulation. Our computational framework is the coupled cluster method which has been presented and discussed in detail in Ref.[1].

We have applied our method to the lattice version of the 2+1 dimensional $SU(2)$ Yang-Mills theory which is the simplest non-trivial non-abelian lattice gauge field theory. We consider this model as an important test case for controlling our computational scheme. Although there is no experiment to compare with, there exist for this model very reliable results for the ground state energy[2] and for glueball masses[3] which allow a critical test of the coupled cluster predictions.

2 Computational Scheme

Within the Hamiltonian formulation, the computational framework is given by the Kogut Susskind theory which has been discussed in detail in Ref.[1]. In summary, the structure is the following:

The Kogut Susskind wave functions $\Psi(U) = \Psi(U_1, .., U_N)$ depend on link variables U_l ($l = 1, .., N$) which are elements of the gauge group $SU(2)$. N is the number of oriented links in a 2-dimensional lattice and is related to the chosen finite volume. Our many-body techniques allow us to perform the final calculations in the infinite volume limit $N \to \infty$.

The idea of the coupled cluster method is to reformulate the eigenvalue problem $H\Psi = E\Psi$ as equations for the ground state correlation function S and for the excitation operators F, following from the ansatzes $\Psi_0(U) = \exp S(U)$ for the ground state and $\Psi(U) = F(U) \exp S(U)$ for excited states.

If $H = 1/(2a)(\sum_{lk} g^2 E_{lk}^2 - 2/g^2 V)$ is the Kogut Susskind Hamiltonian (we use the notations (lk) for link-colour quantum numbers, $(k = 1, ..3)$, E_{lk} for the momentum operators conjugate to U_l and V for the plaquette term.) this yields the non-linear equation

$$\sum_{lk} (S_{lklk} + S_{lk} S_{lk}) - \frac{2}{g^4} V = \frac{2a}{g^2} E_0 \tag{1}$$

for S and the linear equation

$$\sum_{lk}(F_{lklk} + 2S_{lk}F_{lk}) \quad = \quad \frac{2a}{g^2}(E - E_0)F \tag{2}$$

for the excitation operators F. f_{lk} abbreviates "link variable derivatives" of functions $f(U)$:

$$f_{lk} = [E_{lk}, f] \quad , \quad f_{lklk} = [E_{lk}, [E_{lk}, f]] \quad . \tag{3}$$

This (rigorous) form of the eigenvalue problem guarantees manifestly the correct volume dependencies of the ground state energy E_0 and the excitation energies $E - E_0$ (see Ref.[1]).

Within this paper we will restrict ourselves to excitations corresponding to glueballs defining a trivial representation of the lattice Euclidean group (0^+ glueballs with momentum zero). In this case we may write

$$F = \Pi_0 F_{\rm int} \qquad S = \Pi_0 S_{\rm int} \tag{4}$$

where Π_0 is the projection operator on states with vanishing (lattice) momentum, angular momentum and parity plus.

A calculation for general representations of the lattice Euclidean group is possible (see[1]) but technically more involved. As a first test for our method we have chosen the simplest case. Our results encourage us to try the generalization and we are doing this now, but we will defer the presentation of results to a forthcoming publication.

The essential point of the coupled cluster method is to *expand* the intrinsic functions $S_{\rm int}$ and $F_{\rm int}$ with respect to a gauge invariant, linked, standardized basis χ_α in the form

$$S_{\rm int}(U) = \sum_\alpha S_\alpha \chi^\alpha(U) \qquad F_{\rm int}(U) = \sum_\alpha F_\alpha \chi^\alpha(U) \tag{5}$$

and to define approximations by *truncations* of this basis.

Introducing the constant function via $\chi^0 = 1$, the "plaquette function" by putting $\Pi_0\chi^1 = 4V$ and using the strong coupling structure $\sum_{la} \chi^\alpha_{lala} = \epsilon_\alpha \chi^\alpha$, the coupled cluster equations (1) and (2) become equations for the coefficients S_α and F_α

$$\epsilon_\alpha S_\alpha + \sum_\beta C^\beta_\alpha(S) S_\beta \quad = \quad \frac{1}{2g^4}\delta_{\alpha 1} + \frac{aE_0}{4Ng^2}\delta_{\alpha 0} \tag{6}$$

$$\begin{aligned} \epsilon_\alpha F_\alpha + 2\sum_\beta C^\beta_\alpha(S) F_\beta \quad &= \quad \frac{2a}{g^2}(E - E_0)F_\alpha \\ with \qquad C^\beta_\alpha(S) \quad &= \quad \sum_\gamma C^{\gamma\beta}_\alpha S_\gamma \quad . \end{aligned} \tag{7}$$

Here, the coupled cluster matrix elements $C^{\gamma\beta}_\alpha$ are given by

$$C^{\gamma\beta}_\alpha \quad = \quad \frac{1}{2}(\epsilon_\alpha - \epsilon_\beta - \epsilon_\gamma) \sum_u c^{\gamma\beta}_{\alpha u} \tag{8}$$

where the numbers $c^{\beta\gamma}_{\alpha u}$ are related to the action $T(u)$ ($u \in G_E$) of the lattice Euclidean group G_E on the basis χ^α and it's products by

$$\sum_{u \in G_E} \chi^\beta T(u) \chi^\gamma = \sum_{\alpha, u} c^{\beta\gamma}_{\alpha u} T(u) \chi^\alpha \tag{9}$$

Hereby, only those cases have to be considered where the functions χ^β and $T(u)\chi^\gamma$ have a common link variable. Details are given in [1].

The basis χ^α which we use for our calculations is systematically generated from (9) and given by the following prescriptions.

Introduce first a set of "generic" functions $\Lambda_G^{\delta,k}$ given by linked, standardized δ-fold plaquette products. They have the structure

$$\Lambda_G^{\delta,k} = \chi^1 T(u_2(\delta,k))\chi^1 ... T(u_\delta(\delta,k))\chi^1 \ ; \ k = 1,..,n_\delta \tag{10}$$

and we define $n_0 = n_1 = 1$, $\Lambda_G^{0,1} = \chi^0$, $\Lambda_G^{1,1} = \chi^1$.

$\delta(= 0, 1, 2...)$ is the "order" of the function $\Lambda_G^{\delta,k}$.

For our two-dimensional SU(2) case we have $n_\delta = 1, 1, 2, 4, 12, 35, 129$ up to sixth order, respectively. The functions (10) are characterized by simple loop patterns exemplified in [1].

The essential property of this set of functions is that from them one can construct an orthogonal (in the limit $\delta \to \infty$ complete) basis χ^α by acting with suitable Casimir operators of the lattice orthogonal group on $\Lambda_G^{\delta,k}$ and diagonalizing the corresponding Casimir matrices. This basis is then (iteratively) "ordered" by δ because the Casimir operators do not increase δ, i.e. each element χ^α has some order $\delta(\alpha)$.

Up to order six, one obtains in this manner subspaces of the function space of link variables of dimensions $(1, 2, 5, 15, 84, 557, 4972)$, respectively. The relation to the simple generic functions (10), which is set up by diagonalizing the Casimir matrices numerically, allows then to compute also the coefficients $c_{\alpha u}^{\beta\gamma}$ (see [1]).

When constructing the basis χ_α in the way described, one has to take care of possible linear dependencies between the generated functions. In previous investigations [4, 5, 6, 1] this problem was solved by exploiting the Cayley Hamilton relation between matrices. We have used in this connection a much simpler procedure: If a set $f_1(U), .., f_n(U)$ contains only m $(m \leq n)$ independent functions, the matrix $f_i(U^k)(i, k = 1, .., n)$ has for suitable *fixed* variables $(U^1, .., U^n)$ exactly the rank m. Our experience is that statistically chosen variable sets $(U^1, .., U^n) \in SU(2)$ are suitable in this sense. The linear relation between the functions f_i is then easily constructed and the dependent functions can be eliminated.

Having determined in this way all ingredients of the coupled cluster equations (6) and (7), we still have to define a truncation prescription. Up to now, we have used in this connection the proposal of Guo et al[5] which yields actually the simplest set of equations. In this case one puts in the order δ

$$c_{\alpha_3,u}^{\alpha_1,\alpha_2} = 0 \quad \text{for} \quad \delta(\alpha_1) + \delta(\alpha_2) > \delta \tag{11}$$

3 Results and Discussion

3.1 The Ground State

It is a special feature of the Hamiltonian formulation that is provides the energy and the wave function of the vacuum state which is the ground state of the Hamiltonian.

Standard lattice Monte Carlo calculations do not give results for the vacuum energy density. There exist, however, computations within the strong coupling expansion[7] and very reliable Green's function Monte Carlo results[2].

In Figure 1 we compare our coupled cluster results up to sixth order to the results of the other methods. As coupling variable we use the standard expression $\beta = 4/g^2$. We see that we have a good quality and convergence up to $\beta \approx 4$, but our method breaks down for large β where the Green's function Monte Carlo is still valid.

An important feature of our results is that the validity of the coupled cluster method clearly goes beyond the range of the strong coupling expansion which breaks down at $\beta \approx 3$. (Our figure gives the 18th order of strong coupling perturbation theory!) We consider this improvement as a necessary condition for obtaining continuum limit physics.

We should stress that the determination of the vacuum operator $S(U)$ - by solving (6) iteratively - turns out be be much simpler and faster than that of the excitation operator $F(U)$. In this sense we do not see any special difficulty with the fact that the Hamiltonian formulation also involves the determination of the vacuum state. The coupled cluster formulation deals with this problem apparently quite effectively.

Note also that assuming a trivial (strong coupling) vacuum (i.e. $S(U) = 0$) makes no sense within our framework, this would only yield the completely unphysical strong coupling limit spectrum.

In this sense our framework seems to be quite orthogonal to the light front formulation which is based on the assumption that the simplicity of the vacuum could be helpful for the determination of the spectrum.

3.2 The Glueball

The solution of (7) with the lowest eigenvalue has the interpretion of an (approximate) 0^+ glueball mass.

Within the standard lattice Monte Carlo method, a reliable determination of this glueball mass has been achieved by Teper[3].

A comparison to our Hamiltonian results involves a rescaling of the standard "Euclidean" coupling g_E relative to the Hamiltonian coupling g given by

$$\beta_E = \beta + .077 + O(g^2) \tag{12}$$

and a rescaling of the Euclidean masses M_E relative to the Hamiltonian masses M ("velocity of light correction")

$$M_E = (1 + 0.084g^2 + O(g^4))M \tag{13}$$

(See Ref.[8]).

According to Teper's results, the region of asymptotic scaling starts at $\beta_E \approx 3$ for the string tension and for a somewhat larger value of β_E for the glueball masses.

Since our ground state results are reliable up to $\beta \approx 4$, one may hope for a scaling window for our glueball predictions in this range.

A direct comparison to Teper's results is difficult in this β-range because the lowest order relations (12) and (13) are not sufficient for this case. These corrections are large in the strong coupling regime itself because the strong coupling expansions are completely different in the Euclidean and Hamiltonian formalisms; the Euclidean expansion has a logarithmic singularity for $\beta = 0$ which is not present in the Hamiltonian framework.

Therefore, the best test for a window displaying continuum physics is the computation of dimensionless quantities which should become independent of β. Since we have (up to now) only one glueball at our disposal (higher 0^+-glueball masses turn out be be rather unreliable), we have determined within the same coupled cluster approach the "radius" R of the glueball defined by

$$R^2 = \left(\sum_\alpha |F_\alpha|^2 |\chi^\alpha|^2 N(\chi^\alpha) \right) / (\sum_\alpha |F_\alpha|^2 |\chi\alpha|^2) \tag{14}$$

where $N(\chi^\alpha)$ is the number of *different* plaquettes occuring in the corresponding generating function (10). (The Haar measure norms $|\chi^\alpha|$ were calculated using a Monte Carlo method.)

The result for dimensionless quantity RM is given in Figure 2. There is apparently evidence for a scaling window around $\beta \approx 3$.

In Figure 3, we also give our results for the absolute values of the glueball masses. Most interesting is a comparison to the predictions of the (Hamiltonian) strong coupling expansion[7] and of corresponding extrapolations (ELCE-method)[9] which are based on the same Kogut Susskind Hamiltonian.

While the strong coupling expansion breaks down at $\beta \approx 2.1$ (seen in Figure 3 from the 28th order of this expansion) our results show a clear convergence (up to order six) up to $\beta \approx 3$. Thus we are able to make reliable calculations in the range which is beyond, but not too far beyond the radius of convergence of the strong coupling expansion. This is actually the typical range of the reached scaling window in most Monte Carlo lattice QCD calculations.

Comparing to the ELCE method, we see a clear discrepancy already in the range $\beta \approx 3$. The ELCE extrapolation seems to overestimate the mass values and it is difficult to understand how this result will converge to the asymptotic glueball mass, given by Teper as $M(g \to 0)/2g^2 = 3.2$ with a 1% error.

The coupled cluster method, on the other hand, appears to give a prediction that does not contradict to this continuum limit value, although a direct comparison to Teper's result is hampered by the lack of good knowledge of the rescaling corrections between Hamiltonian and Lagrangian formulation for $\beta \approx 3$.

Summarizing we conclude that the coupled cluster method appears to be able to produce predictions within Hamiltonian lattice field theories in a range of the couplings where asymptotic scaling is already valid. Hereby, the simultaneous computation of the vacuum state poses no special difficulty, it may be reliably determined in the region where observables show a scaling behaviour (and beyond, i.e. up to $\beta \approx 4$ in our case). Also the numerical effort is much less for the vacuum than for the diagonalization problem of the glueball masses.

Clearly, a more decisive test of our computational framework will be our predictions for glueball masses with more general (lattice) angular momentum quantum numbers and the comparison of the emerging mass ratios to those of Teper. Calculations in this direction are now being undertaken.

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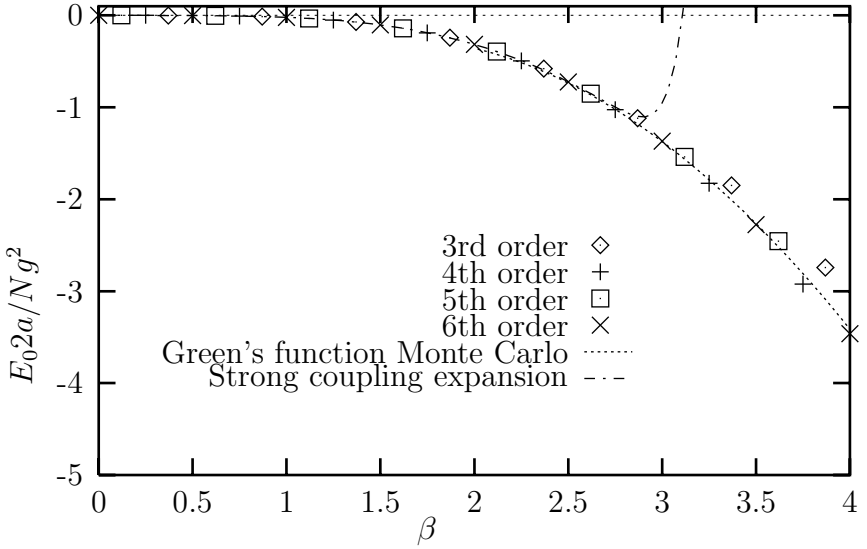
Figure Captions

Figure 1: Coupled cluster predictions of this work for the ground state energy density in 3rd, 4th, 5th and 6th order, compared with the 18th order of the strong coupling expansion [7] and the results of the Green's function Monte Carlo method [2]. The upper part magnifies the low β range of the results to show the quality of the convergence.

Figure 2: The dimensionless product of glueball mass and radius MR in 4th, 5th and 6th order.

Figure 3: Mass M of the 0^+ glueball in 4th (- - - -), 5th (- - -) and 6th (——) order in comparison with the 28th order of the strong coupling expansion [7] (lower dot-dashed line) and an extrapolation using the ELCE method [7] (diamonds with error bars). The upper dot-dashed line gives the Monte Carlo results of Teper [3] using a lowest order Euclidean-Hamiltonian rescaling given by eqs. (12) and (13). The Monte Carlo errors (1-3%) are small compared to the uncertainty of these rescaling corrections (see text) and are therefore not indicated in the figure. The insert shows a magnification of the crucial part between $\beta = 2$ and $\beta = 3$.

Figure 1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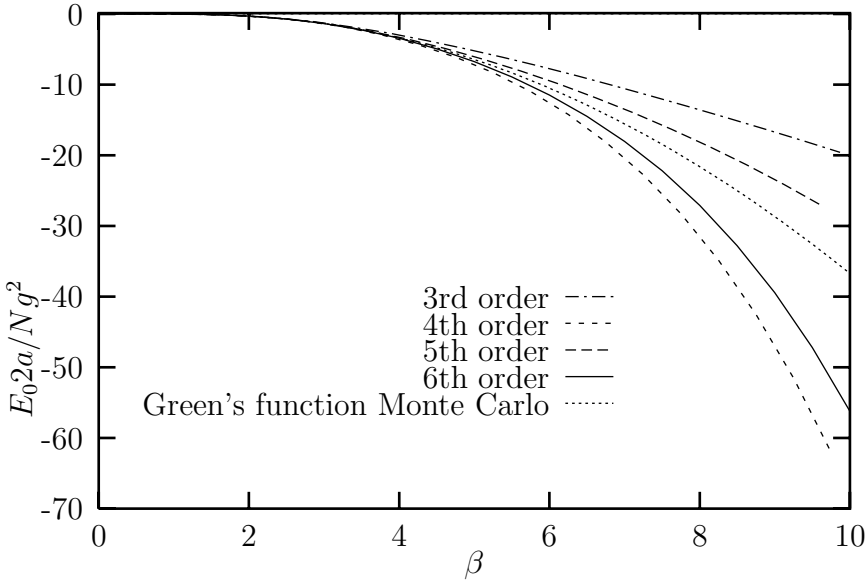
Green's function Monte Carlo
Strong coupling expansion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Green's function Monte Carlo

Figure 2

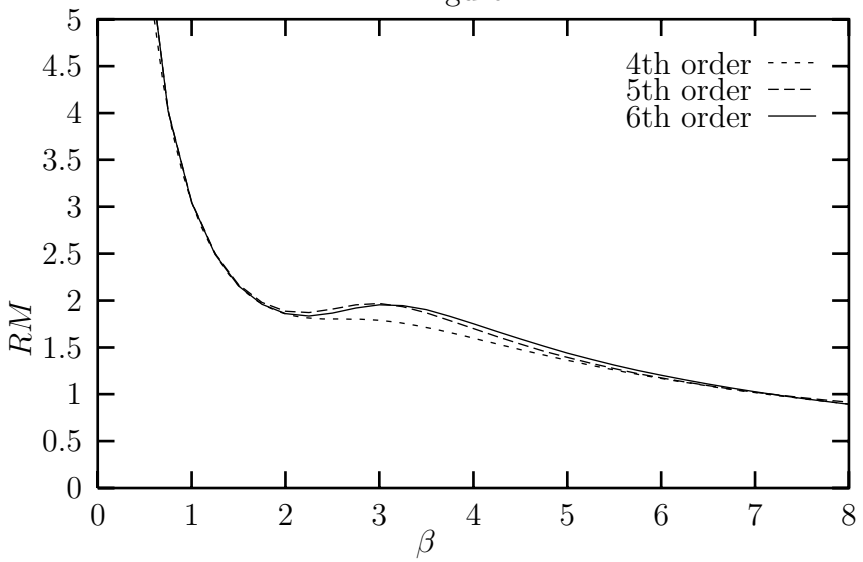
4th order
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Figure 3